Finite difference method for solving crack problems in a functionally graded material

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Abstract
A linear elastic two-dimensional formulation for functionally graded materials is presented. The two-dimensional equilibrium equations and boundary conditions in an orthogonal curvilinear coordinate system are written explicitly. The finite difference technique is used to solve the above formulation. The solution technique is verified by solving two test problems, in which the material is graded horizontally and vertically. The results are compared to analytical results and have very good agreement. The solution technique is then applied to solve a long layer containing an edge crack in which it is assumed that the Young’s modulus varies continuously along its width. The problem is solved for two loading conditions: tension and bending. The mode I stress intensity factor is extracted by applying three methods: \( J \) line and two versions of a modified conservative \( J \) integral for graded materials. All three methods provide similar results, which are in excellent agreement with the semi-analytical results in the literature. These results demonstrate the applicability of the finite difference technique for solving crack problems in functionally graded materials.

Keywords
Functionally graded materials, finite difference method, edge crack, stress intensity factor

1. Introduction
The development of functionally graded materials (FGMs) offers new alternatives for engineering design. These materials enable engineers to tailor the material properties of a part, and not just its dimensions or shape, to meet design objectives and constraints. FGMs are characterized by gradual spatial changes in their properties due to changes in their composition or structure. Usually, they are composites, but they can also be monolithic materials that do not contain well-defined boundaries or interfaces between their various regions. The smooth transition of properties avoids high interlaminar stresses and delamination, which affects laminated composites. Because of the improved mechanical behavior, these materials are increasingly used in a variety of engineering applications. Developments of the processing techniques of FGMs are constantly made, and the development of three-dimensional (3D) printing, which has the capability of extruding two different filaments from one nozzle, will further increase FGM usability.

Cracking and fracture of such materials can occur and have been extensively investigated, as described in the literature. Numerical models, such as integral equations, boundary elements, and the finite element method, have been used to investigate FGMs. A comprehensive review of the various analytical and numerical methods employed to study the static, dynamic, and stability behaviors of FGM plates is given in Swaminathan et al. All possible evaluation techniques of such materials (aimed at enriching numerical capability) should be investigated. Having more than one numerical technique that converges to the same result, especially in problems that do not have analytical solutions, is an advantage.

In this investigation, the finite difference (FD) method is applied to solve crack problems in FGMs. The accuracy that can be achieved in the calculation of stress intensity factors (SIFs) is checked for a mode I crack problem in FGMs. This technique is simple and easy to apply. It was successfully applied to investigate 3D delamination, and bimaterial crack problems with contact and friction.
between the crack faces of linear homogeneous elastic materials.

Three post-processing routines for SIF determination were implemented. In Section 2, the linear elastic two-dimensional formulation for FGMs is presented. The equilibrium equations and the boundary conditions are transformed to an orthogonal curvilinear coordinate system and written explicitly. A verification of the technique is performed in Section 3, wherein two test cases with analytical solutions are solved successfully. Section 4 addresses the edge crack problem. The first part of this section details the three methods used to extract the SIF and their application in the FD technique. Then, the edge crack problem is introduced and solved. The manuscript ends with a summary and conclusions.

## 2. The FD formulation

The two-dimensional displacement equilibrium equations in an \( x, y \) Cartesian coordinate system for linear elastic FGMs with small strains, neglecting body forces, are given by:

\[
(1 + K) U_{,xx} + U_{,yy} + KV_{,xy} + \frac{a_{xx}}{G} U_{,x} + \frac{b_{xy}}{G} V_{,y} + \frac{G_{,y}}{G} (U_{,y} + V_{,x}) = 0
\]

(1)

\[
(1 + K) V_{,yy} + V_{,xx} + KV_{,xy} + \frac{a_{yy}}{G} V_{,y} + \frac{b_{xy}}{G} U_{,x} + \frac{G_{,x}}{G} (U_{,x} + V_{,y}) = 0
\]

(2)

where a comma represents differentiation. The last four terms in Equations (1) and (2), which include first derivative of the displacements, do not appear for homogeneous materials. The material coefficients for the plane stress and plane strain conditions are detailed in Table 1. Young’s modulus \( E \) and Poisson’s ratio \( \nu \) are functions of location: \( E = f_1(x, y) \) and \( \nu = f_2(x, y) \).

Stresses are related to displacements through Hooke’s law as follows:

\[
\sigma_{xx} = a_{xx} U_{,x} + b_{xy} V_{,y} + a U_{,x} + b V_{,y}
\]

(3)

\[
\sigma_{yy} = a_{yy} V_{,y} + b_{xx} U_{,x} + a U_{,y} + b V_{,x}
\]

(4)

\[
\sigma_{xy} = 2G_{xy} = G(U_{,y} + V_{,x})
\]

(5)

For plane strain:

\[
\sigma_{xx} = \nu (\sigma_{xx} + \sigma_{yy}) = h(U_{,x} + V_{,y})
\]

(6)

For generality, the derivatives in the above equations are transformed to a curvilinear coordinate system \( (\xi, \eta) \) by means of the chain rule. Such a transformation enables solutions in any curvilinear coordinates, e.g., rectangular or cylindrical coordinates, and increases the capability of refining the mesh at areas of stress concentration, as will be shown in subsequent research. Transformed equilibrium Equations (1) and (2) in the \( x \)- and \( y \)-directions, respectively, are given by:

\[
(b^1 + b^2 + b^3) U_{,\xi} + (b^1 + b^2 + b^3) U_{,\eta} + b^1 U_{,\xi\eta} + b^2 U_{,\xi\eta} + \left( c^1 + c^2 + c^3 \right) V_{,\xi} + \left( c^1 + c^2 + c^3 \right) V_{,\eta} + c^1 V_{,\xi\eta} + c^2 V_{,\xi\eta} = 0
\]

(7)

\[
(d^1 + d^2 + d^3) V_{,\xi} + (d^1 + d^2 + d^3) V_{,\eta} + d^1 U_{,\xi\eta} + d^2 U_{,\xi\eta} + \left( c^1 + c^2 + c^3 \right) U_{,\xi} + \left( c^1 + c^2 + c^3 \right) U_{,\eta} + c^1 U_{,\xi\eta} + c^2 U_{,\xi\eta} + c^3 U_{,\xi\eta} + c^4 U_{,\xi\eta} = 0
\]

(8)

Coefficients \( b_i, c_i, d_i \) \( i = 1 \ldots 5 \) and \( b^7, c^7, d^7 \) \( i = 1 \ldots 2 \) depend on material properties and mixed derivatives of both coordinates systems at an interior point of the FD mesh at which the equilibrium equations are imposed. Equations (7) and (8) are the same equations as those for homogeneous materials. The only difference from the homogeneous case lies in coefficients \( b_i, c_i, d_i \) \( i = 1 \ldots 2 \). These coefficients appear because of the contribution of the four terms with a first derivative of the displacements in equilibrium Equations (1) and (2). These terms appear because the material is nonhomogeneous. Coefficients \( b^i, c^i, d^i \) \( i = 1 \ldots 2 \) are detailed in Appendix A.

The stresses corresponding to Equations (3)–(6) are transformed to a curvilinear coordinate system via:

\[
\sigma_{\xi\xi} = c_{11} U_{,\xi} + c_{12} U_{,\eta} + c_{13} V_{,\xi} + c_{14} V_{,\eta}
\]

(9)

\[
\sigma_{\eta\eta} = c_{21} U_{,\xi} + c_{22} U_{,\eta} + c_{23} V_{,\xi} + c_{24} V_{,\eta}
\]

(10)

\[
\sigma_{\xi\eta} = c_{31} U_{,\xi} + c_{32} U_{,\eta} + c_{33} V_{,\xi} + c_{34} V_{,\eta}
\]

(11)

Coefficients \( c_{ij}, i = 1 \ldots 3, j = 1 \ldots 4 \) depend on the mixed derivatives of both coordinate systems at the point of application of the boundary conditions, are given explicitly in Appendix A.

The tractions on the curvilinear boundary are given by:

\[
T_i = \sigma_i n_i ; \quad i = \xi, \eta ; \quad j = \xi, \eta
\]

(12)
where \( T_i \) are the tractions, \( \sigma_j \) are the stresses in Equations (9)–(12), and \( \eta_j \) are the components of the outward unit normal to the boundary in the \((\xi, \eta)\) directions. The displacements on the curvilinear boundary are given by:

\[
U = c_{41} U_\xi + c_{42} U_\eta
\]

(13)

\[
V = c_{51} U_\xi + c_{52} U_\eta
\]

(14)

where \( U_\xi \) and \( U_\eta \) are displacements in the curvilinear directions. Coefficients \( c_{ij} \); \( i = 4 \cdots 5; j = 1 \cdots 2 \) are also presented in Appendix A.

The physical domain \((x, y)\) is mapped into a Cartesian numerical domain \((\xi, \eta)\) in which \(1 \leq \xi \leq 2 \) and \(1 \leq \eta \leq 2\), as shown in Figure 1. Equations (7)–(14) are used in the numerical domain. Mapping is performed using polynomial transformations given by23:

\[
x = \sum_{i=0}^{M} s_i \xi^i
\]

(15)

\[
y = \sum_{i=0}^{N} t_i \eta^i
\]

(16)

where \( s_i \); \( i = 0 \cdots M \) and \( t_i \); \( i = 0 \cdots N \) are constants. Proper choice of the coefficients allows for mesh refinement at the crack tip in the physical domain. The FD mesh in the numerical domain has a constant mesh size \( h \) and \( k \) in the \( \xi \) and \( \eta \) directions, respectively. The application of the FD technique is the same as in previous studies.23–25

Central FD formulas (stencils), as detailed in Appendix B, are substituted into Equations (7)–(11). The residuals of all the difference equations approach zero as \( h^2 \to 0 \) and \( k^2 \to 0 \). At each mesh point, there are two unknowns, that is, displacements \( U \) and \( V \). For each mesh point in the interior, there are two equilibrium equations (Equations (7) and (8)). For each imaginary mesh point, there are two boundary conditions applied on the adjacent boundary point (yellow). These boundary conditions can be displacements, tractions or mixed traction-displacements. The values of the four imaginary corner mesh points are determined via linear extrapolation. The algebraic linear system of equations resulting from the substitution of the stencils into Equations (7)–(14) is programmed in Fortran 90 (Intel Fortran compiler, Intel Parallel Studio XE 2013, Intel Corporation) and solved with a Fortran library subroutine, Y12MAF (http://www.netlib.org/y12m/doc) that solves sparse systems of algebraic linear equations by Gaussian elimination.

3. Verification of the FD solution

Four test problems were solved to verify the accuracy of the FD technique. These test cases include material gradation but not stress concentration or stress singularities. The four simple test problems are described in Cartesian coordinates and have closed-form solutions for plane stress non-homogeneous isotropic materials.26 The problems consist of uniaxial tension \( T \) in the \( x \) direction of a square plate with \(0 \leq x \leq 1\) and \(0 \leq y \leq 1\) and nonhomogeneous properties \( E(x, y) \). In the first two problems, Young’s modulus varies only along the \( x \) direction as \( E(x) = \frac{E_0}{1 + kx} \) with \( k = -0.5, 5 \). The analytical solution for the displacements is found to be as follows:

\[
u_{d}(x, y) = \frac{T}{E_0} \left(1 + \hat{k}x\right) y
\]

(17)

\[
u_{d}(x, y) = -\nu \frac{T}{E_0} \left(1 + \hat{k}x\right) y
\]

(18)

In the third and fourth problems, Young’s modulus varies only along the \( y \) direction as \( E(y) = \frac{E_0}{1 + ky} \) with \( \hat{k} = -0.5, 5 \). The analytical solution for the displacement is also found in Sadd:

\[
u_{d}(x, y) = \frac{T}{E_0} \left(1 + \hat{k}y\right) x
\]

(19)

\[
u_{d}(x, y) = -\nu \frac{T}{E_0} \left(y + \hat{k} \left(\frac{y^2}{2} + \frac{x^2}{2}\right)\right)
\]

(20)

The geometry, load, and a color map that shows the \( x \) distribution of \( E(x) \) for \( k = -0.5 \) are shown in Figure 2(a). The variations of \( E \) along \( x \) for \( k = -0.5 \) and 5 are shown in Figure 2(b). In the numerical analyses, \( E_0 = 1, \nu = 0.3 \), and plane stress conditions prevail.

The boundary conditions applied numerically for all test cases are:

At \( x = 0 \): \( T_x(0,y) = -\sigma_{xx}(0,y) = -1, T_y(0,y) = \sigma_{yy}(0,y) = 0 \)

At \( x = 1 \): \( T_x(0,y) = \sigma_{xx}(1,y) = 1, T_y(1,y) = \sigma_{yy}(0,y) = 0 \)

At \( y = 0 \): \( u(x, 0) = u_d(x, 0) \) and \( v(x, 0) = v_d(x, 0) \), where \( u_d \) and \( v_d \) are detailed in Equations (17)–(20).
At $y = 1$: $T_s(x, 1) = \sigma_{yy}(x, 1) = 0$ and $T_y(x, 1) = \sigma_{yy}(x, 1) = 0$

3.1. Test case 1: material gradation along $x$ with $k = -0.5$ and $k = 5$

An equal mesh size was used with application of linear transformations (Equations (15) and (16)), where $x = -1 + \xi$ and $y = -1 + \eta$. The mesh size was increased gradually until very good agreement between the numerical and analytical displacements of the upper free face at $y = 1$ was obtained. The initial number of mesh points was $20 \times 20$ and the final number was $150 \times 150$ points. These numbers correspond to mesh sizes in the transformed coordinates $(\xi, \eta)$ of $h = k = 1/18 = 0.0556$ and $h = k = 1/148 = 0.0068$, respectively.

Figure 3(a) shows displacements $u$ and $v$ on the upper free face ($y = 1$) for the case where $k = 5$. The results of $u_n$ and $v_n$ (obtained with a $20 \times 20$ coarse mesh and a $150 \times 150$ fine mesh, respectively) are presented along with the analytical displacements. For the coarse mesh, results for $v_n$ differ significantly from the analytical results, while results for $u_n$ are quite accurate. For the fine mesh ($150 \times 150$), very good agreement can be observed.

The relative error of the displacements is calculated via:

$$error_u [%] = 100 \frac{u_n(x_i) - u_d(x_i)}{u_{dmax}}$$

$$error_v [%] = 100 \frac{v_n(x_i) - v_d(x_i)}{v_{dmax}}$$

where $x_i$ is a mesh point along the boundary and $u_{dmax}$ is the maximum displacement $u$ at $(x, y) = (1, 1)$. The error for the case where $k = -0.5$ indicates that, for the coarse mesh, the error in $u_n$ is less than 7% (except from the corner point at $(x, y) = (0, 1)$). For the fine mesh, the maximum error drops below 1.1%.

The results for the case where $k = 5$ indicate that the usage of the $20 \times 20$ coarse mesh results in considerably large errors; the error of $u_n$ reaches 15% and the error of $v_n$ reaches 5%. Increasing the number of mesh points to $120 \times 120$ reduces the errors significantly. The maximum error in $u_n$ is 2.7%, while the errors in $v_n$ are all below 1%. A further increase of the number of points to $150 \times 150$ reduces the maximum error in $u_n$ to 2.2%, while the errors of $v_n$ remain below 1%.

3.2. Test case 2: material gradation along $y$ with $k = -0.5$ and $k = 5.0$

An equal size coarse mesh of $20 \times 20$ points was used to solve this problem. The results of the coarse mesh were verified by using a finer mesh of $40 \times 40$ points. The accuracy of the displacements on the upper free face at $y = 1$ was checked with the aid of Equations (21) and (22). Figure 4 shows the analytical displacements on the upper face alongside the numerical ones. Figure 4(a) shows the case where $k = -0.5$ and Figure 4(b) shows the case where $k = 5.0$. Excellent agreement can be observed for both cases (even for the coarse mesh). The error is less than 0.01% (except from the corner point at $(x, y) = (0, 1)$, at which it is 3%). The finer mesh with $40 \times 40$ points confirms the results obtained by the coarse mesh.

3.3. Summary

The results of these test cases verify the accuracy of the FD method presented. The results demonstrate convergence, that is, the smaller the mesh size, the more accurate the results. A considerably fine mesh is needed to obtain...
accurate results when the direction of the material gradation and the load coincide (test cases 1 and 2). A coarse mesh is needed when the material gradation and the load direction do not coincide (test cases 3 and 4). A crack problem with stress singularity will be examined next.

4. The edge crack problem

The aim of this section is to demonstrate the application of the FD technique for solving crack problems in FGMs. An edge crack is a common type of structural damage, and therefore, it is very useful to investigate its fracture characteristics. Many studies have investigated the mode I SIFs of cracks in FGMs in unbounded materials. Erdogan and Wu solved an edge problem in a bounded region composed of a long strip of finite width in an FGM material. They assumed that the shear modulus is graded exponentially along the width, and the Poisson’s ratio remains constant. The purpose of this investigation is to first reproduce the results obtained by Erdogan and Wu, using the FD method.

The methods used to extract the mode I SIF from the FD results are summarized and explained, and then the problem and results are introduced.

4.1. Methods for calculating the mode I SIF

Three methods have been used in this investigation to extract the mode I SIF from the FD results. They all involve the $J$ integral. These methods have been applied in other investigations for finite elements results. They are briefly introduced here and their application for FD results is explained.

4.1.1. A path-dependent $J$ line. The line integral $J$ is as follows:

$$ J_0 = \int_{\Gamma_0} \left( W_{n_0} - T_i \frac{\partial u_i}{\partial x} \right) d\Gamma $$

where path $\Gamma_0$ is any path that starts on the bottom crack face, surrounds the crack tip counterclockwise and ends on the upper crack face. An example of such a path is shown in Figure 5(a). The displacements are $u_i$, the tractions along this path are $T_i = \sigma_{ij}n_j$ and $n_j$ are the components of the outward normal to the path. The strain energy density is given by $W = \frac{1}{2} \sigma_{ij}e_{ij}$. A circular path for computing $J$ line $\Gamma_0$, which surrounds the crack tip at $r = \epsilon$, is also shown schematically in Figure 5(a). The value of this integral when $\epsilon \rightarrow 0$ is $J_0$.

Eischen has shown that for FGMs:

$$ J_\epsilon = \int_{\Gamma_\epsilon} \left( W_{n_\epsilon} - T_i \frac{\partial u_i}{\partial x} \right) d\Gamma = \frac{K^2_1 + K^2_2}{E'_{np}} $$

Figure 3. Comparison between analytical and numerical results of the displacements at $y = 1$. The material is graded in the $x$ direction. (a) $k = -0.5$. (b) $k = 5$.

Figure 4. Comparison between analytic and numerical results of the displacements at $y = 1$. The material is graded in the $y$ direction. (a) $k = -0.5$. (b) $k = 5$.
where

\[
\begin{align*}
E_{\text{tip}} = E_{\text{tip}} & \quad \text{plane stress} \\
E = \frac{E_{\text{tip}}}{1 - \nu_{\text{tip}}} & \quad \text{plane strain}
\end{align*}
\]  

(25)

The subscript “tip” denotes the values at \((x_{\text{tip}}, y_{\text{tip}})\)—the coordinates of the crack tip. Once \(J_e\) is known, the mode I SIF can be calculated by:

\[
K_i = \sqrt{J_e E}
\]  

(26)

A typical rectangular FD mesh on a cracked component is shown in Figure 6. The crack tip lies at \((x, y) = (x_{\text{tip}}, 0)\). Because of symmetry, only the region above the crack is considered. Four typical lines for calculating \(J_e\) are also illustrated. The path starts at \((x_R, 0)\), climbs vertically to \((x_L, y_T)\), continues horizontally to \((x_L, y_r)\) and ends on the crack face at \((x_L, 0)\). The integral \(\Delta J_e\) of Equation (24) is calculated by summing each interior point \(i\) that lies on the path to approximate the integral as \(J_0 \approx \sum_{i} \Delta J_e d\Gamma_i\), where \(d\Gamma_i = dx_i\) or \(d\Gamma_i = dy_i\).

Integration is performed symmetrically on both sides of the crack tip, that is, \(|x_{\text{tip}} - x_R| = |x_{\text{tip}} - x_L|\), and when \(|x_{\text{tip}} - x_R| = \rho \to 0\), the area encircled by the path approaches zero and the value of \(J\) corresponds to \(J_e\). The integral is calculated over many paths and the results are curve-fitted by polynomial \(J_0 \approx f(\rho)\). Here \(J_e\) is calculated by taking the limit \(\rho \to 0\).

4.1.2. A conservative \(J\) integral. A closed path for calculating the \(J\) line is shown in Figure 6(b). The path surrounds area \(A\) and excludes the crack tip. The value of the \(J\) integral can be written as:

\[
J_0 - J_e = \oint_{\Gamma_0 + \Gamma_1 + \Gamma_2 + \Gamma_3} \left( W_{n_x} - T_{i} \frac{\partial n_i}{\partial x} \right) \, dV
\]  

(27)

Note that along crack faces \(\Gamma_1\) and \(\Gamma_2\), the integrand vanishes, and along \(\Gamma_3\), normal \(n\) points inward. By invoking the divergence theorem, Equation (27) can be written as:

\[
J_0 - J_e = \int_{A} \left[ \frac{\partial W}{\partial x} - \frac{\partial}{\partial x_i} \left( \sigma_{ij} \frac{\partial n_j}{\partial x} \right) \right] \, dA
\]  

(28)

The derivative of the strain energy is as follows:

\[
\frac{\partial W}{\partial x} = \frac{\partial W}{\partial n_{ij}} \frac{\partial n_{ij}}{\partial x} + \frac{\partial W}{\partial E} \frac{\partial E}{\partial x} + \frac{\partial W}{\partial \nu} \frac{\partial \nu}{\partial x} = \sigma_{ij} \frac{\partial n_{ij}}{\partial x} + \frac{\partial W}{\partial E} \frac{\partial E}{\partial x} + \frac{\partial W}{\partial \nu} \frac{\partial \nu}{\partial x}
\]  

(29)

For homogeneous materials where \(\frac{\partial W}{\partial x} = \frac{\partial W}{\partial \nu} = 0\), the integral of Equation (28) equals 0 and \(J_0 - J_e = 0\); therefore, \(J_0 = J_e\) and \(J\) integral is path independent.

For FGMs, it becomes as follows:

\[
J_0 - J_e = \int_{A} \left[ \frac{\partial W}{\partial n_{ij}} \frac{\partial n_{ij}}{\partial x} + \frac{\partial W}{\partial E} \frac{\partial E}{\partial x} + \frac{\partial W}{\partial \nu} \frac{\partial \nu}{\partial x} \right] \, dA = \int_{A} \frac{\partial W^*}{\partial x} \, dA
\]  

(30)

Therefore:

\[
J_e = J_0 - \int_{A} \frac{\partial W^*}{\partial x} \, dA
\]  

(31)
Equation (31) is path independent but includes both line integral $J_0$ and an area integral. The calculation of $J_0$ is performed as explained in Section 4.1.1. The area encircled by path $\Gamma_2$ is shown in Figure 6. The integrand in Equation (31) is calculated within each interior point $i$ and summed as $\int_{A} \frac{\partial W}{\partial x} \, dA \cong \sum_{i} \left( \frac{\partial W}{\partial x} \right)_{i} dA_i$, where $dA_i = dx_i dy_i$ is the area associated with internal mesh point $i$.

4.1.3. A conservative $J$ area integral. If we have a continuous function $q$ with values of zero along $\Gamma_0$ and 1 along $\Gamma_1$, we can multiply both sides of Equation (28) to get the following:

$$ (J_0 - J_c)q = -J_c = \oint_{\Gamma_0 + \Gamma_1 + \Gamma_2 + \Gamma_3} \left( Wn_x - T_i \frac{\partial u_i}{\partial x} \right) q \, d\Gamma $$

(32)

Invoking the divergence theorem on the right side of Equation (32):

$$ -J_c = \int_{A} \left( W\delta_{ij} - T_i \frac{\partial u_i}{\partial x_j} \right) \frac{\partial q}{\partial x_j} + \frac{\partial W}{\partial x} - \frac{\partial}{\partial x_j} \left( \sigma_{ij} \frac{\partial u_i}{\partial x_j} \right) q \, dA $$

(33)

After some algebraic manipulation, the integral may be written as:

$$ J_c = \int_{A} \sigma_{ij} \frac{\partial u_i}{\partial x_j} \, dA - \int_{A} \frac{\partial W}{\partial x} q \, dA $$

(34)

The implementation of the area integral is the same as explained in Section 4.1.2.

Function $q$ was chosen as:

$$ q(x, y) = \frac{(x - x_L)(x - x_R)(y - y_T)}{(x_{tip} - x_L)(x_{tip} - x_R)(y_{tip} - y_T)} $$

(35)

The shape of $q$ for $(x_{tip}, y_{tip}) = (0.5, 0.0)$ and $x_L = 0.2$, $x_R = 0.8$ and $y_T = 0.3$ is shown in Figure 7. Figure 7(a) shows a 3D image of $q$, and Figure 7(b) shows a color contour map. It is clearly observed that $q$ has values of zero for $\Gamma_0$ and 1 for $\Gamma_1$ (crack tip).

4.2. The edge crack problem

Figure 8 shows the geometry of the edge crack problem. The height is eight times the width ($\frac{H}{W} = 4$) to mimic a long strip, and length $a$ of the crack is half width $W$ ($\frac{a}{W} = 0.5$). Two loading conditions were analyzed: tension and bending. The applied boundary conditions are:

$$ T_x(x, H) = -T_y(x, -H) = \sigma_0 \quad \text{tension} $$

$$ T_y(x, H) = -T_x(x, -H) = \sigma_0 \left( 1 - \frac{2x}{W} \right) \quad \text{bending} $$

The Young’s modulus of an FGM is assumed to vary exponentially as $E(x) = E_1 e^{\beta x}$, and the Poisson’s ratio is constant. Modulus variation $E(x)$ is characterized by two parameters: $E_1 = E(0)$ and $E_2 = E(W)$. Parameter $\beta = \frac{1}{W} \log \left( \frac{E_2}{E_1} \right)$. Plane strain conditions are assumed.

Because of symmetry, only the upper half is analyzed numerically. Free-face conditions ($\sigma_{xy} = \sigma_{yx} = 0$) are applied for the crack face. The component is fixed in space by constraining the crack tip: $u = v = 0$. Symmetry conditions $v = 0$, $\frac{\partial u}{\partial y} = 0$ are applied all along the symmetry line.

In the analyses, the following parameters were used: $W = 1, a = 0.5, H = 4, E_1 = 1000, \nu = 0.3, \text{ and } \sigma_0 = 10$. A non-uniform mesh, which is dense at the region of the crack tip, is used. A mesh with 160 mesh points in the $x$ direction and 120 mesh points in the $y$ direction is used. Figure 9 shows a transformation from the physical domain...
to the numerical domain (Equations (15) and (16)). The mesh is dense where the derivatives $\frac{\partial \sigma_0}{\partial x}$ and $\frac{\partial \sigma_0}{\partial y}$ are small.

The mesh in physical domain ($x, y$) in the neighborhood of the crack tip, which is located at $(x_0, y_0) = (0.5, 0)$, is shown in Figure 10. The mesh sizes near the crack tip, $D_x$ and $D_y$, are shown in the figure. The size of $D_x$ and $D_y$ shown is ~0.11% the crack length. The first three nearest paths for calculating $J$ are shown in the figure.

4.3. Edge crack results

The edge crack problem was solved for two loading conditions: tension and bending. The following cases were solved: $E_2/E_1 = 0$:

First, an example of the SIF calculation under tension loading for case $E_2/E_1 = 0$ is shown. Figure 11 shows the results of 75 $J_0$, $J_e$, and $J_{area}$ calculations along 75 paths and the areas they enclosed. The first three paths are shown in Figure 10. The path are symmetric in the $x$ direction, i.e., $x_R - x_{tip} \simeq x_{tip} - x_L$, which means that when $x_R - x_{tip} \rightarrow 0$, area $A$ around the crack tip is enclosed by such path $A \rightarrow 0$.

It is clearly observed that $J_0$ is path dependent, but its value for $x_R - x_{tip} \rightarrow 0$ converges to the expected value. Here $J_e$ (Section 4.1.2) and $J_{area}$ (Section 4.1.3) are path independent, and a constant value is obtained for $J$ for any distance from the crack tip. Some small divergence is observed near the crack tip because FD does not accurately predict the field very close to the crack tip. $J_0$ is approximated by a fourth-order polynomial, from which the limit when $x_R - x_{tip} \rightarrow 0$ is extracted: $\frac{J_0}{\sigma_0} = 1.195$.

Averaging the values of $J_e$ and $J_{area}$ in $0.2 < \frac{x_R - x_{tip}}{a} < 0.6$ yields the normalized values 1.1144 and 1.1196, respectively. Mode I SIF is calculated using Equations (25) and (26) as the following values (which are normalized by $s_0$): 3.636, 3.511, and 3.519 for $J_0$, $J_e$, and $J_{area}$, respectively.

Table 2 summarizes the results and compares them to the results obtained in (10) using $\text{dif} = \frac{K_f - K_{FD}}{K_f} \times 100\%$, where $K_f$ is the result obtained in Erdogan and Wu. Table 2 and 3 summarize the results for tension and bending loading, respectively. The cases solved numerically are: $E_2/E_1 = 0.1, 0.2, 0.5, 1, 2, 5, 10$. Very good agreement between the FD results and those of Erdogan.
and Wu,\textsuperscript{10} the analytic results for $E_2/E_1 = 1$ having been taken from Anderson,\textsuperscript{28} can be observed in Tables 2 and 3; differences are less than 1\% except for three cases (out of 16): $E_2/E_1 = 0.1$ in tension loading, and $E_2/E_1 = 0.1$ and 0.2 for bending loading. The maximum difference in tension is 1.97\%. In bending loading, the maximum difference is 2.63\%.

Overall, the good agreement with the results of Erdogan and Wu,\textsuperscript{10} and the agreement between the three post-processors applied on the FD results, confirm the reliability to the FD method.

5. Summary and conclusions

A linear elastic two-dimensional formulation for FGMs in the $(x, y)$ coordinate system is presented. The formulation of the equilibrium equations and boundary conditions were transformed into the orthogonal curvilinear coordinate system and were written explicitly. The FD technique was used to solve the above formulation using the Fortran 90 programming language. The solution technique was first verified by solving two test problems, in which the material was graded horizontally and vertically. A comparison with the analytical results showed very good agreement. The solution technique was then applied to solve an edge crack in a long layer. The Young’s modulus of the layer was assumed to vary exponentially with width. The problem was solved for two loading conditions: tension and bending. To extract the mode I SIF from the FD results, three post-processor routines were developed and applied: (a) $J$ line close to the crack tip, (b) a conservative $J$ integral for FGM, and (c) a conservative $J$ area integral for FGMs. All three methods provided similar results and were in excellent agreement with the semi-analytical results of Erdogan and Wu.\textsuperscript{10} The analytic results for $E_2/E_1 = 1$ are from Anderson.\textsuperscript{28}

<table>
<thead>
<tr>
<th>$E_2/E_1$</th>
<th>$J_0$</th>
<th>$J_\varepsilon$</th>
<th>$J_{area}$</th>
<th>Erdogan and Wu\textsuperscript{10}</th>
</tr>
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<tbody>
<tr>
<td>0.1</td>
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<td>3.5014</td>
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<td>1.32</td>
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<td>3.301</td>
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<td>3.3266</td>
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<td>0.77</td>
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<tr>
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<td>0.15</td>
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Figure 10. Mesh in $(x, y)$ near the crack tip.

Figure 11. Results of calculations of $J_0$, $J_\varepsilon$, and $J_{area}$ versus the distance of the path from the crack tip.
Erdogan and Wu. This agreement indicates that the obtained stress and strain fields are accurate, which also leads to accurate results for the $J$ integrals. These results demonstrate the applicability of the FD technique for solving crack problems in FGMs. The accurate SIF results encourage further study in the field of fracture mechanics associated with contact and friction, as in previous studies.

Acknowledgement
The assistance and support of Professor Daniel Rittel is greatly appreciated.

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References
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Appendix A

Coefficients for equilibrium equations and boundary conditions in the curvilinear orthogonal coordinate system

Let (ξ, η) be a curvilinear orthogonal coordinate system in which x = X(ξ, η) and y = Y(ξ, η). Let f(x, y) = f[X(ξ, η), Y(ξ, η)] be any function on that two-dimensional region. Using the “chain rule of derivatives,” the first derivative can be written as:

\[ f_x = f_x x_x + f_y y_x \quad i = \xi \eta \]  
(A1)

The second derivative is:

\[ f_{xx} = f_{xx x_x x_x} + f_{xy x_x y_x} + f_{yx y_x y_x} + f_{yy y_y y_x} \] 
\[ + f_{xy y_x y_y} + f_{yx y_y y_y} + f_{yy y_y y_y} \]  
(A2)

Here, ij = (ξξ, ηη, ξη).

Solving the two equations (Equations (A1)) yields the following:

\[ f_{x} = \frac{1}{J} (y, y, \xi, \eta - y, \xi, \eta) \]  
\[ f_{y} = \frac{1}{J} (-x, y, \xi, \eta + x, y, \xi, \eta) \]  
\[ f_{\xi} = \frac{1}{J} \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{yx} & \alpha_{yy} \end{bmatrix} \begin{bmatrix} f_{x} \\ f_{y} \\ f_{\xi} \\ f_{\eta} \end{bmatrix} \]  
\[ = \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{yx} & \alpha_{yy} \end{bmatrix} \begin{bmatrix} \frac{x}{J^2} \\ \frac{y}{J^2} \\ \frac{-2y y y_x x_x}{J^2} \\ \frac{2x y y_x y_y}{J^2} \end{bmatrix} \]  
(A3)

where the Jacobian is:

\[ J = x, y, \eta - y, \xi x, \eta \]  
\[ f_{\eta} = \frac{1}{J} \begin{bmatrix} \beta_{xx} & \beta_{xy} & \beta_{yx} & \beta_{yy} \end{bmatrix} \begin{bmatrix} f_{x} \\ f_{y} \\ f_{\xi} \\ f_{\eta} \end{bmatrix} \]  
\[ = \begin{bmatrix} \beta_{xx} & \beta_{xy} & \beta_{yx} & \beta_{yy} \end{bmatrix} \begin{bmatrix} \frac{x}{J^2} \\ \frac{y}{J^2} \\ \frac{-2y y y_x x_x}{J^2} \\ \frac{2x y y_x y_y}{J^2} \end{bmatrix} \]  
\[ c_{xx} = \frac{y^2}{J^2} \]  
\[ c_{yy} = \frac{x^2}{J^2} \]  
\[ c_{xy} = \frac{-2x y y_x y_y}{J^2} \]  
\[ c_{yx} = \frac{-2x y x_x x_x}{J^2} \]  
\( J \) is the Jacobian of the transformation.

The solution of the three equations (Equation (A2)) can be written as:

\[ \begin{bmatrix} f_{xx} \\ f_{yy} \\ f_{xy} \end{bmatrix} = \begin{bmatrix} a_{xx} & a_{xy} & a_{yx} & a_{yy} \\ b_{xx} & b_{xy} & b_{yx} & b_{yy} \\ c_{xx} & c_{xy} & c_{yx} & c_{yy} \end{bmatrix} \begin{bmatrix} f_{x} \\ f_{y} \\ f_{\xi} \\ f_{\eta} \end{bmatrix} \]  
\[ = \begin{bmatrix} a_{xx} & a_{xy} & a_{yx} & a_{yy} \\ b_{xx} & b_{xy} & b_{yx} & b_{yy} \\ c_{xx} & c_{xy} & c_{yx} & c_{yy} \end{bmatrix} \begin{bmatrix} \frac{x}{J^2} \\ \frac{y}{J^2} \\ \frac{-2y y y_x x_x}{J^2} \\ \frac{2x y y_x y_y}{J^2} \end{bmatrix} \]  
(A5)
Substitution of Equations (A6)–(A20) in the equilibrium equations yields the coefficients of the equilibrium equations (Equations (7) and (8)). Here, \( a, b, k, \) and \( G \) are material parameters detailed in Table 1:

\[
d_{yy} = \frac{x^2}{J^2} \tag{A14}
\]

\[
e_{yy} = \frac{-2x\xi^2\eta}{J^2} \tag{A15}
\]

\[
a_{yy} = \frac{1}{J^2} \left( y_1 \eta y_1 \xi + y_2 \eta y_2 \xi - x_1 \eta x_1 \xi - x_2 \eta x_2 \xi \right) \tag{A16}
\]

\[
b_{yy} = \frac{1}{J^2} \left( -y_1 \eta y_1 \xi - y_2 \eta y_2 \xi + x_1 \eta x_1 \xi + x_2 \eta x_2 \xi \right) \tag{A17}
\]

\[
c_{yy} = \frac{-y_1 \eta y_1 \xi}{J^2} \tag{A18}
\]

\[
d_{yy} = \frac{-y_1 \eta y_1 \xi}{J^2} \tag{A19}
\]

\[
e_{yy} = \frac{x_1 \eta y_1 \xi + x_2 \eta y_2 \xi}{J^2} \tag{A20}
\]

The coefficients for Equations (9)-(11) for application of traction boundary condition are as follows:

\[
c_{11} = \frac{a_1 (1-\nu) h_1^2 \eta \xi}{(1+\nu) (1-2\nu)} E \tag{A44}
\]

\[
c_{12} = \frac{a_2 h_2^2 \eta \xi}{(1+\nu) (1-2\nu)} E \tag{A45}
\]

\[
c_{13} = \frac{a_3 (1-\nu) h_3^2 \eta \xi}{(1+\nu) (1-2\nu)} E \tag{A46}
\]

\[
c_{14} = \frac{a_4 h_4^2 \eta \xi}{(1+\nu) (1-2\nu)} E \tag{A47}
\]

\[
c_{21} = \frac{a_5 h_5^2 \eta \xi}{(1+\nu) (1-2\nu)} E \tag{A48}
\]

\[
c_{22} = \frac{a_6 (1-\nu) h_6^2 \eta \xi}{(1+\nu) (1-2\nu)} E \tag{A49}
\]

\[
c_{23} = \frac{a_7 (1-\nu) h_7^2 \eta \xi}{(1+\nu) (1-2\nu)} E \tag{A50}
\]

\[
c_{24} = \frac{a_8 h_8^2 \eta \xi}{(1+\nu) (1-2\nu)} E \tag{A51}
\]

where

\[
h_\xi = \frac{1}{\sqrt{(x, \xi)^2 + (y, \xi)^2}} \tag{A56}
\]
For plane strain, \(a_1 = a_2 = a_3 = 1\), and for plane stress, 
\[ a_1 = \frac{1 - 2\nu}{(1 - \nu)^2}, \quad a_2 = \frac{1 - 2\nu}{(1 - \nu)} \quad \text{and} \quad a_3 = 0. \]

The coefficients in Equations (13) and (14) for application of the displacement boundary conditions are given by:

\[ c_{41} = h_x x, \xi \]  
\[ c_{42} = h_x y, \eta \]  
\[ c_{51} = h_y x, \xi \]  
\[ c_{52} = h_y y, \eta \]  

**Appendix B**

**Central finite difference formulas**

Interior points:

\[
\begin{align*}
U_{x, x} &= \frac{U(i+1,j) - U(i-1,j)}{2h} \\
U_{x, y} &= \frac{U(i+1,j) - U(i,j-1)}{2h} \\
U_{x, xx} &= \frac{U(i-1,j) - 2U(i,j) + U(i+1,j)}{h^2} \\
U_{x, yy} &= \frac{U(i+1,j) - 2U(i,j) + U(i-1,j)}{h^2} \\
U_{x, xy} &= \frac{U(i+1,j+1) + U(i-1,j-1) - U(i+1,j-1) - U(i-1,j+1)}{4h^2}
\end{align*}
\]

Boundary points: \( n_x = \pm 1 \)

\[
\begin{align*}
U_{x, x} &= \frac{1}{2} \left( \frac{U(i+1,j) - U(i,j)}{2h} + \frac{U(i+1,j) - U(i,j-1)}{2h} \right) \\
U_{x, y} &= \frac{1}{2} \left( \frac{U(i,j+1) - U(i,j-1)}{2h} + \frac{U(i,j) - U(i,j-1)}{2h} \right)
\end{align*}
\]

Boundary points: \( n_y = \pm 1 \)

\[
\begin{align*}
U_{x, x} &= \frac{1}{2} \left( \frac{U(i+1,j) - U(i,j)}{2h} + \frac{U(i,j+1) - U(i,j-1)}{2h} \right) \\
U_{x, y} &= \frac{1}{2} \left( \frac{U(i,j) - U(i,j-1)}{2h} + \frac{U(i,j) - U(i,j+1)}{2h} \right)
\end{align*}
\]

**Author biography**

**Avraham Dorogoy** gained his BSc in Mechanical Engineering (Cum Laude) in 1982, and MSc in 1985 at Technion, Haifa, Israel. Avraham holds a PhD in Mechanical Engineering since 2002 from Tel Aviv University, Israel. Avraham returned to the Faculty of Mechanical Engineering, Technion, Haifa, Israel at 2003 and is currently a senior research engineer and a senior adjunct lecturer in “fracture mechanics.”